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On the set of lines stabbing two convex polygons : An analysis using geometric algebra and leading to a minimal representation in the Plücker space

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Abstract

Polygon-to-polygon visibility can be computed using CSG operations in the Plücker space. In this context, a polytope is built to represent the set of lines stabbing two convex polygons. The polytope representation is conservative since only its intersection with the so-called Plücker quadric is the real set of stabbing lines. To avoid unecessary computations, it is usefull to build a polytope as close as possible to the set of stabbing lines. In this technical report, we propose a polytope representation for the lines stabbing two convex polygons, and we proove that this polytope is minimal, *i.e.* it is the smallest polytope including the set of stabbing lines.

Contents

1 Mathematical preliminaries

1.1 A short introduction to geometric algebras

Geometric Algebras are Grassmann algebras with an additionnal geometric product defined over Clifford algebras. This geometric product explanation is not necessary to the demonstration presented in this paper. As a consequence we refer the interested reader to [1] for further details on this product.

In this section we recall how to build a Grassmann algebra and what are its properties. Next, we explain the relation between Plücker space and a subspace of a particular Grassmann Algebra.

An algebraic structure can be build from a vector space and a product of vectors which is distributive over the vector space addition [3]. Geometric algebra relies on a multivector space, denoted $\mathscr{G}_n$. First, we present this space.

Let $\mathscr{E}$ be a n-dimensional vector space. We denote '$\wedge$' the product on $\mathscr{E}$, which is linear, associative and antisymetric. $\mathscr{E}$ is the *geometric* space, it represents the geometry. Let a k-vector be a multivector of rank k given as the product of k vectors from $\mathscr{E}$. We denote $\mathscr{G}_n^k$ the k-vector space, which has a vector space structure. As an example, let a and b be two vectors in $\mathscr{E}$. $a \wedge b$ is a 2-vector and satisfies $a \wedge b = -b \wedge a$. If $b = a$ then $a \wedge a = 0$ since the product is antisymetric.

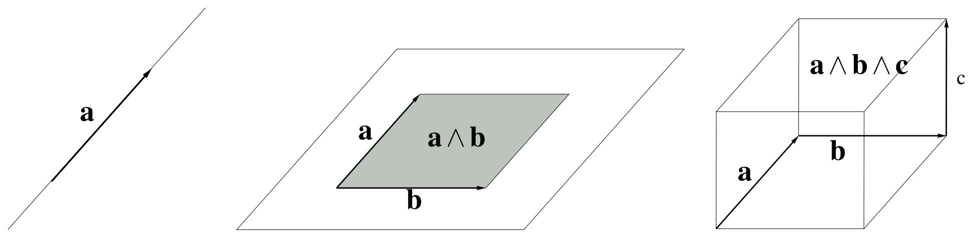
The '$\wedge$' product is also called the exterior product. It is the product over the Grassmann algebra (or exterior algebra [4]). From this product and a vector space, the multivector space can be defined as a k-vectors sum, $0 \leq k \leq n$. If $k = 1$, we can build the 1-vector space, which is isomorph to $\mathscr{E}$. If $k = 0$, it defines the 0-vector space, it can be considered as the space of scalars.

The exterior product allows the algebraic representation of all the oriented vector subspaces of $\mathscr{E}$. The geometric interpretation of a 1-vector is an oriented line segment. The geometric interpretation of a 2-vector is an oriented area element. An an example, the 2-vector $\mathbf{a} \wedge \mathbf{b}$ represents the oriented area element from the plane defined by the 1-vector a and b. The figure 1 gives an illustration. The geometric interpretation of a k-vector is an oriented segment in k-dimensional vector subspace. Its norm is the volume defined by the k 1-vectors spanning the k-vector.

1.2 Plücker coordinates of lines as a special case of Grassmann algebra

Let $\mathbb{P}^3$ be the projective space isomorphic to vector lines in $\mathbb{R}^4$. That is the space of equivalent classes where a vector a is equivalent to λa, for all a in $\mathbb{R}^4$ and λ in $\mathbb{R} \setminus \{0\}$. This space is often used in computer graphics to represent points and points at infinity in $\mathbb{R}^3$, by vectors in $\mathbb{R}^4$ also called homogeneous points or homogeneous coordinates.

If we consider the Grassmann algebra $\mathscr{G}_4$ obtained from $\mathbb{R}^4$, a homogeneous point become a (homogeneous) 1-vector in $\mathscr{G}_4$. Let P and Q be two homogeneous 1-vectors in $\mathscr{G}_4$. The 2-vector $P \wedge Q$ caracterizes all the vectors X of $\mathbb{R}^4$ (*i.e.* 1-vectors of $\mathscr{G}_4$) that satisfy $X \wedge P \wedge Q = 0$ which is a two dimensional sub-space of $\mathbb{R}^4$. Homogeneous vectors in this 2-space are exactly the points that lie on the line from P to Q.



(a) 1-vector, determine a unique line

(b) 2-vector, determine a unique plane

(c) 3-vector, determine a unique three dimensional space

Figure 1: Interpretation of k-vectors in $\mathscr{E}$

Actually, it is easy to see that every line in $\mathbb{R}^3$ can be represented by a 2-subspace of $\mathbb{R}^4$. Conversely, every 2-subspace in $\mathbb{R}^4$, projected on a hyperplane out of the origin (vector $\overrightarrow{0}$ of $\mathbb{R}^4$ don't lie on this hyperplane) represents a line of $\mathbb{R}^3$. Yet, all 2-subspaces in $\mathbb{R}^4$ is represented in $\mathscr{G}_4$ by a 2-blade, that is to say a decomposable 2-vector. This 2-blade is in fact the exterior product of two homogeneous 1-vectors representing any two points of the given line.

Equivalence with Plücker coordinates of lines in $\mathbb{R}^3$

We can show that the Plücker coordinates $(Q-P : P \times Q)$ of the line (PQ) are the coordinates of the 2-vector $P \wedge Q$ in $\mathscr{G}_4$ in a particular base of 2-vectors in $\mathscr{G}_4$.

Let $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ be an orthonormal base of $\mathbb{R}^4 = \varepsilon_0 \oplus \mathbb{R}^3$. Let $(\varepsilon_0 \wedge \varepsilon_1, \varepsilon_0 \wedge \varepsilon_2, \varepsilon_0 \wedge \varepsilon_3, \varepsilon_2 \wedge \varepsilon_3, \varepsilon_3 \wedge \varepsilon_1, \varepsilon_1 \wedge \varepsilon_2)$ be a base of 2-vectors in $\mathscr{G}_4$. Let $P = p_0\varepsilon_0 + p_1\varepsilon_1 + p_2\varepsilon_2 + p_3\varepsilon_3$ and $Q = q_0\varepsilon_0 + q_1\varepsilon_1 + q_2\varepsilon_2 + q_3\varepsilon_3$ two vector in $\mathbb{R}^4$. The 2-vector $P \wedge Q$ is equal to :

$$\begin{aligned}
P \wedge Q &= (p_0\varepsilon_0 + p_1\varepsilon_1 + p_2\varepsilon_2 + p_3\varepsilon_3) \wedge (q_0\varepsilon_0 + q_1\varepsilon_1 + q_2\varepsilon_2 + q_3\varepsilon_3) \\
&= \textstyle\sum_{i,j \in \{0,1,2,3\}, i \neq j} (p_i q_j) \varepsilon_i \wedge \varepsilon_j \\
&= (q_1 p_0 - p_1 q_0)\varepsilon_0 \wedge \varepsilon_1 + (q_2 p_0 - p_2 q_0)\varepsilon_0 \wedge \varepsilon_2 + (q_3 p_0 - p_3 q_0)\varepsilon_0 \wedge \varepsilon_3 + \\
&\quad (q_3 p_2 - p_3 q_2)\varepsilon_2 \wedge \varepsilon_3 + (q_1 p_3 - p_1 q_3)\varepsilon_3 \wedge \varepsilon_1 + (q_2 p_1 - p_2 q_1)\varepsilon_1 \wedge \varepsilon_2
\end{aligned}$$

Assuming P and Q are homogeneous (that is $p_0 = q_0 = 1$), we have :

$$\begin{aligned}
p \wedge q &= (q_1 - p_1)\varepsilon_0 \wedge \varepsilon_1 + (q_2 - p_2)\varepsilon_0 \wedge \varepsilon_2 + (q_3 - p_3)\varepsilon_0 \wedge \varepsilon_3 + \\
&\quad (q_3 p_2 - p_3 q_2)\varepsilon_2 \wedge \varepsilon_3 + (q_1 p_3 - p_1 q_3)\varepsilon_3 \wedge \varepsilon_1 + (q_2 p_1 - p_2 q_1)\varepsilon_1 \wedge \varepsilon_2
\end{aligned}$$

And we retrieve exactly the known Plücker coordinates.

2 On the lines stabbing two convex polygons

2.1 Problem statement

To minimize the number of effective intersections during the polygon to polygon visibility computation process, it is usefull to have the minimal representation of the set of lines stabbing two convex polygons.

Until now, the proposed solutions where approximations of this representation. Moreover, no method can measure the closeness of these approximations with the minimal one.

We propose here to give an answer to this problem by prooving the construction of the minimal representation and its conditions of validity.

2.2 The minimal polytope

Theorem

The set of lines stabbing two convex polygons **A** *and* **B** *in* $\mathbb{R}^3$ *is the intersection of the Grassmannian* $G^{\mathbb{R}}(4,2)$ *with the convex hull of the lines going through one vertex of* **A** *and one vertex of* **B** *if and only if the support planes of* **A** *and* **B** *don't intersect on* **A** *or* **B**.

Proof

We recall that the lines which go by one vertex of **A** and one vertex of **B** are called ESL (for Extremal Stabbing Lines). The theorem to proove can be expressed with no constraint on **A** and **B** :

$$D \in V_{\mathbf{A},\mathbf{B}} \Leftrightarrow D^* \in G^{\mathbb{R}}(4,2) \text{ and } D^* \in Conv_{esl}$$

where D is an affine line of the 3D space and D^* its representation by a 2-vector in $\mathscr{G}_4$, lying on the Grassmannian $G^{\mathbb{R}}(4,2)$, $V_{\mathbf{A},\mathbf{B}}$ is the set of lines stabbing A and B and $Conv_{esl}$ the convex hull of the ESL.

The demonstration is decomposed in two parts for the two implications.

$D \in V_{\mathbf{A},\mathbf{B}} \Rightarrow D^* \in G^{\mathbb{R}}(4,2)$ **and** $D^* \in Conv_{esl}$

The first implication $D \in V_{\mathbf{A},\mathbf{B}} \Rightarrow D^* \in G^{\mathbb{R}}(4,2)$ and $D^* \in Conv_{esl}$ is really simple to prove, in the design of exterior algebra :

All the lines stabbing **A**, of vertices $(a_i)_{i=1..n}$ and **B**, of vertices $(b_j)_{j=1..m}$ can be represented by a 2-vector obtained from the exterior product of one point on **A** and one point on **B**. Since B is a decomposable 2-vector of $\mathscr{G}_4$, B is on the Grassmannian $G^{\mathbb{R}}(4,2)$.

Let a and b, respectively two any points on **A** and **B**. Two sequences $(\varepsilon_i^a)_{i=1\cdots n} \in \mathbb{R}^{n^+}$ and $(\varepsilon_j^b)_{j=1\cdots m} \in \mathbb{R}^{m^+}$ such that $a = \sum_{i=1}^{n} \varepsilon_i^a b_i$ and $b = \sum_{j=1}^{m} \varepsilon_j^b b_j$ can be determined. It is useless to require that the sum of coefficients must be equal to 1 since we are in equivalence classes and we don't care about wich representative of these classes we consider.

Thus, $B = s \dot{\wedge} c$ is equal to :

$$B = \sum_{i,j} \varepsilon_i^s \varepsilon_j^c s_i \dot{\wedge} c_j$$

Yet $\varepsilon_i^s \varepsilon_j^c \geq 0$, deducing that the 2-vector B belongs to the convex hull $Conv_{esl}$ of the ESL, which prove the first implication.

$D^* \in G^{\mathbb{R}}(4,2)$ **and** $D^* \in Conv_{esl} \Rightarrow D \in V_{\mathbf{A},\mathbf{B}}$

The converse is more difficult to prove and is not verified in some degenerate cases. We will consider the general case appart from those degeneracy.

The general case

We are in the general case, when the faces **A** and **B** don't intersect. In other words, the intersection line of the support planes of **A** and **B** don't pass through **A** or **B**.

Let three points a on **A**, b on **B** (not on the boundary of **B**) and e, on the support plane of **B** but not on **B**. The segment $[c\,e]$ intersect the boundary of **B** on a unique point i, distinct from c and e. This intersection i is whether on one vertex of **B** (that is the intersection of two edges of **B**) or on one of its edges. In all cases, let E be one edge of **B** on which i lie. Clearly, the line represented by $a \dot{\wedge} e$ is not a line stabbing **B**.

It is easy to verify, for the hyperplane $\mathscr{H}_{E^*}$ hortogonal to E^* in $\mathbb{R}^{3,3}$ and correctly oriented relatively to **A** and **B**, that all the ESL $\mathbf{e}_i$ verify $(H_A \wedge \mathbf{e}_i)^* \geq 0$. It is trivial that All points in the convex hull $Conv_{esl}$ also verify this property.

We have the relation :

$$e = (1+\lambda)i - \lambda c, \lambda \in \mathbb{R}_*^+$$

And so :

$$s \dot{\wedge} e = (1+\lambda) s \dot{\wedge} i - \lambda s \dot{\wedge} c$$

Yet, $(1+\lambda) a \dot{\wedge} i$ and $\lambda a \dot{\wedge} b$ are both in $Conv_{esl}$. In the same way, $(H_A \wedge (a \dot{\wedge} i))^* = 0$ and $(H_A \wedge (a \dot{\wedge} b))^* > 0$.
This gives :

$$(H_A \wedge (s \dot{\wedge} e))^* = -\lambda (H_A \wedge (s \dot{\wedge} c))^* < 0$$

It can be deduced that $a \dot{\wedge} e$ is not in the convex hull $Conv_{esl}$. The same process can be achieved for any point out of **A**.

Notice also that every 2-vectors of $\mathscr{G}_4$ which is not decomposable do not represent an affine line is not a visibility line, this proves $\neg(D \in V_{\mathbf{S},\mathbf{C}}) \Rightarrow \neg(D_{plg} \in G^{\mathbb{R}}(4,2)$ and $D_{plg} \in Conv_{esl})$ which is equivalent to the second implication :

$$D_{plg} \in G^{\mathbb{R}}(4,2) \text{ and } D_{plg} \in Conv_{esl} \Rightarrow D \in V_{\mathbf{S},\mathbf{C}}$$

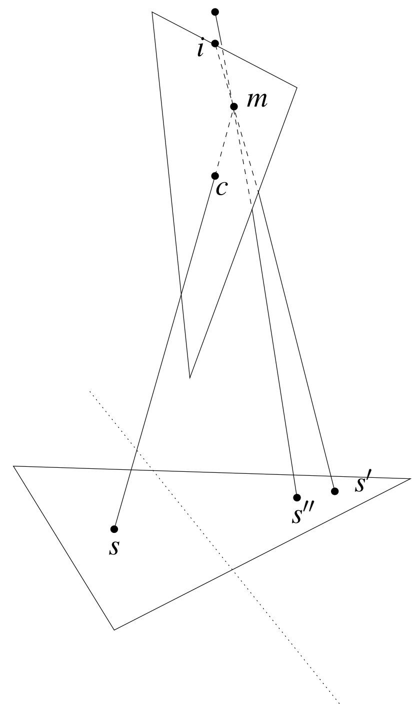


Figure 2: One degenerate case where the second implication is not verified

Degenerate cases
The degenerate cases occure when the support plane of one face among **A** and **B** (eventually the support plane of the two faces), intersect the other face. We only consider the case where the support plane of **B** intersect the face **A** since the symetric case has the same properties. As for the case the two faces both intersect, it is a particular case of the two previous one. Thus, showing the second implication is false in the first case show it is false in the second one.

Here, we can consider two "subcases". The first occures when this intersection separate **A** in two faces (**A** is split by the plane). The second occures when the intersection line of the two support plane of **A** and **B** is on one edge of **A** (**A** is tangent to the plane).

Let us consider the first subcase for the moment. Let two points a and a' of **A**, each on different side relatively to the support plane of **B**. Let a point i, on an edge of **B** which do not lie on the plane determined by the three points s, s' and i. Let b a point on **B**, lying on the plane determined by the triplet (a, a', i).

The two lines (a, b) and (a', i) are both visibility lines (implying $a \dot{\wedge} b$ and $a' \dot{\wedge} i$ are in $Conv_{esl}$), sequent on the point m (figure 2). We have :

$$\begin{cases} m = s + \lambda(i - s) & \lambda \in \mathbb{R}_*^+ \\ m = s' + \lambda'(c - s') & \lambda' \in \mathbb{R}_*^+ \end{cases}$$

Which gives :

$$\begin{cases} s \dot{\wedge} i = \frac{1}{\lambda}\, s \dot{\wedge} m \\ s' \dot{\wedge} c = \frac{1}{\lambda'}\, s' \dot{\wedge} m \end{cases}$$

Let $s'' = \gamma s + (1-\gamma)s',\ \gamma \in \mathbb{R}_*^+$. Thus, $\lambda\gamma\, s \wedge i + \lambda'(1-\gamma)\, s' \wedge c = s'' \wedge m$ is in $Conv_{esl}$ since it is a sum of two points in $Conv_{esl}$. Yet, it is possible, as illustrated on figure 2, to find values for γ such that $s'' \wedge m$ is not in the set of visibility. We conclude that the converse implication is not verified in this first degenerate case.

Concerning the second subcase, if we consider the line that do not stab the edge of **A** at the intersection with the support plane of **B**, we retrieve the general case, and the second implication is true for these lines.

As for the lines that stab the intersected edge of **A** and the face **B**, they all lie in the support plane of **B**, and this case is similar to the second subcase but in two dimensions. As we have seen, if one support line of any edge of **B** intersect the concerned edge of **A**, the second implication is not verified.

In practice, it is always possible to construct the polytope $Conv_{esl}$ if this case occures, it is just required that the lines stabbing the line at intersection of the two support planes of **A** and **B** are ignored from the visibility, which is not inconsistent since those lines are tangent to **B** (or **A** in the symetric case) and lie on the boundary of $Conv_{esl}$.

We can also notice that these lines correspond to points at infinity, according to the definition of the projective hyperplane in [2]. But in previous methods, all points are projected on this hyperplane, which forbid representation of points at infinity. Furthermore, since they do not have the definition of the minimal polytope as the convex hull of the ESL representation, their polytope is not consistent in this case. So our method, contrary to the previous ones, can deal with all degenerate cases (if any degenerate case occure, we can always bring back to this valid degenerate case by split the intersected faces in two distinct faces) without *ad-hoc* strategies.

References

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